Exercise 2.11 Solution

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Let C_{BR} be the true direction cosine matrix of the attitude with respect to the reference frame and $C_{B'R}$ be the estimated direction cosine matrix of the attitude. Then, the error direction cosine matrix, $C_{BB'}$, is given by

$$C_{BB'} = C_{BR}\,C_{RB'} = C_{BR}\,C_{B'R}^T$$

As the direction matrix is a function of the quaternion, see (2.37) in the book, the same operation with the corresponding quaternions is given by

$$\delta\mathbf{q}_{BB'} = \mathbf{q}_{BR} \otimes \mathbf{q}_{RB'} = \mathbf{q}_{BR} \otimes \mathbf{q}_{B'R}^{-1} \tag{1}$$

where $\otimes$ is the quaternion product defined by

$$\mathbf{q}^{(1)} \otimes \mathbf{q}^{(2)} = \begin{bmatrix} q_4^{(1)}\mathbf{q}_{13}^{(2)} + q_4^{(2)}\mathbf{q}_{13}^{(1)} - [\mathbf{q}_{13}^{(1)}\times]\mathbf{q}_{13}^{(2)} \\ q_4^{(1)}q_4^{(2)} - \left(\mathbf{q}_{13}^{(1)}\right)^T\mathbf{q}_{13}^{(2)} \end{bmatrix}$$

$$\mathbf{q}^{(i)} = \begin{bmatrix} \mathbf{q}_{13}^{(i)} \\ q_4^{(i)} \end{bmatrix} = \begin{bmatrix} q_1^{(i)} & q_2^{(i)} & q_3^{(i)} & q_4^{(i)} \end{bmatrix}^T$$

for i be the positive integers, and the inverse quaternion is given by

$$\mathbf{q}^{-1} = \begin{bmatrix} -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix}^T$$

i.e., the same angle rotation with the opposite direction to the one of $\mathbf{q} = [q_1, q_2, q_3, q_4]^T$.

Take the time derivative of (1)

$$\delta\dot{\mathbf{q}}_{BB'} = \dot{\mathbf{q}}_{BR} \otimes \mathbf{q}_{B'R}^{-1} + \mathbf{q}_{BR} \otimes \dot{\mathbf{q}}_{B'R}^{-1} \tag{2}$$

where

$$\dot{\mathbf{q}}_{BR} = \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q}_{BR}$$

To obtain the time derivative of $\mathbf{q}_{B'R}^{-1}$ in (1), take the time derivative of the following equation:

$$\mathbf{q}_{B'R} \otimes \mathbf{q}_{B'R}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

as follows:

$$\dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1} + \mathbf{q}_{B'R} \otimes \dot{\mathbf{q}}_{B'R}^{-1} = \mathbf{0}$$

where $\mathbf{0}$ is the 4×1 zero vector. Apply $\mathbf{q}_{RB'} \otimes$ both sides

$$\mathbf{q}_{RB'} \otimes \dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1} + \mathbf{q}_{RB'} \otimes \mathbf{q}_{B'R} \otimes \dot{\mathbf{q}}_{B'R}^{-1} = \mathbf{0}$$

and it becomes

$$\mathbf{q}_{RB'} \otimes \dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1} + \dot{\mathbf{q}}_{B'R}^{-1} = \mathbf{0} \Rightarrow \dot{\mathbf{q}}_{B'R}^{-1} = -\mathbf{q}_{RB'} \otimes \dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1}$$

where

$$\dot{\mathbf{q}}_{B'R} = \frac{1}{2}\Omega(\hat{\boldsymbol{\omega}})\mathbf{q}_{B'R} \tag{3}$$

and

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega} - \delta\boldsymbol{\omega}$$

Obtain the expression of the right-hand side of

$$\dot{\mathbf{q}}_{B'R}^{-1} = -\mathbf{q}_{RB'} \otimes \dot{\mathbf{q}}_{B'R} \otimes \mathbf{q}_{B'R}^{-1}$$

using the symbolic calculations in MATLAB or python (See matlab_exercise_2_11.py or python_exercise_2_11.py) as follows:

$$\begin{aligned}
\dot{\mathbf{q}}_{B'R}^{-1} &= \frac{1}{2}\begin{bmatrix} -q_2\hat{\omega}_3 + q_3\hat{\omega}_2 - q_4\hat{\omega}_1 \\ -q_3\hat{\omega}_1 + q_1\hat{\omega}_3 - q_4\hat{\omega}_2 \\ -q_1\hat{\omega}_2 + q_2\hat{\omega}_1 - q_4\hat{\omega}_3 \\ -q_1\hat{\omega}_1 - q_2\hat{\omega}_2 - q_3\hat{\omega}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \hat{\omega}_3 & -\hat{\omega}_2 & -\hat{\omega}_1 \\ -\hat{\omega}_3 & 0 & \hat{\omega}_1 & -\hat{\omega}_2 \\ \hat{\omega}_2 & -\hat{\omega}_1 & 0 & -\hat{\omega}_3 \\ \hat{\omega}_1 & \hat{\omega}_2 & \hat{\omega}_3 & 0 \end{bmatrix}\begin{bmatrix} -q_1 \\ -q_2 \\ -q_3 \\ q_4 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} -[\hat{\boldsymbol{\omega}}\times] & -\hat{\boldsymbol{\omega}} \\ \hat{\boldsymbol{\omega}}^T & 0 \end{bmatrix}\mathbf{q}_{B'R}^{-1} = \frac{1}{2}\mho(\hat{\boldsymbol{\omega}})\mathbf{q}_{B'R}^{-1}
\end{aligned}$$

where the subscript $B'R$ is omitted for the notational simplicity, or it can be written as

$$\begin{aligned}
\dot{\mathbf{q}}_{B'R}^{-1} &= \frac{1}{2}\begin{bmatrix} -[\hat{\boldsymbol{\omega}}\times] & -\hat{\boldsymbol{\omega}} \\ \hat{\boldsymbol{\omega}}^T & 0 \end{bmatrix}\mathbf{q}_{B'R}^{-1} = \frac{1}{2}\begin{bmatrix} [\left(\mathbf{q}_{B'R}^{-1}\right)_{13}\times]\hat{\boldsymbol{\omega}} - \hat{\boldsymbol{\omega}}\left(q_{B'R}^{-1}\right)_4 \\ \hat{\boldsymbol{\omega}}^T\left(\mathbf{q}_{B'R}^{-1}\right)_{13} \end{bmatrix} \\
&= -\frac{1}{2}\mathbf{q}_{B'R}^{-1} \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix}
\end{aligned} \tag{4}$$

Substitute (3) and (4) into (2)

$$\begin{aligned}\delta\dot{\mathbf{q}}_{BB'} &= \left[\frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q}_{BR}\right]\otimes\mathbf{q}_{B'R}^{-1} - \frac{1}{2}\mathbf{q}_{BR}\otimes\left\{\mathbf{q}_{B'R}^{-1}\otimes\begin{bmatrix}\hat{\boldsymbol{\omega}}\\0\end{bmatrix}\right\}\\ &= \frac{1}{2}\Omega(\boldsymbol{\omega})\left[\mathbf{q}_{BR}\otimes\mathbf{q}_{B'R}^{-1}\right] - \frac{1}{2}\left[\mathbf{q}_{BR}\otimes\mathbf{q}_{B'R}^{-1}\right]\otimes\begin{bmatrix}\hat{\boldsymbol{\omega}}\\0\end{bmatrix}\end{aligned} \tag{5}$$

where the following two equalities

$$\left[\frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q}_{BR}\right]\otimes\mathbf{q}_{B'R}^{-1} = \frac{1}{2}\Omega(\boldsymbol{\omega})\left[\mathbf{q}_{BR}\otimes\mathbf{q}_{B'R}^{-1}\right] \tag{6}$$

$$\mathbf{q}_{BR}\otimes\left\{\mathbf{q}_{B'R}^{-1}\otimes\begin{bmatrix}\hat{\boldsymbol{\omega}}\\0\end{bmatrix}\right\} = \left[\mathbf{q}_{BR}\otimes\mathbf{q}_{B'R}^{-1}\right]\otimes\begin{bmatrix}\hat{\boldsymbol{\omega}}\\0\end{bmatrix} \tag{7}$$

are proved in matlab_exercise_2_11.m or python_exercise_2_11.py.

Equation (5) is further expanded as follows:

$$\begin{aligned}\delta\dot{\mathbf{q}}_{BB'} &= \frac{1}{2}\Omega(\boldsymbol{\omega})\delta\mathbf{q}_{BB'} - \frac{1}{2}\delta\mathbf{q}_{BB'}\otimes\begin{bmatrix}\hat{\boldsymbol{\omega}}\\0\end{bmatrix} = \frac{1}{2}\Omega(\boldsymbol{\omega})\delta\mathbf{q}_{BB'} + \frac{1}{2}\mho(\hat{\boldsymbol{\omega}})\mathbf{q}_{BB'}\\ &= \frac{1}{2}\left[\Omega(\hat{\boldsymbol{\omega}}+\delta\boldsymbol{\omega}) + \mho(\hat{\boldsymbol{\omega}})\right]\mathbf{q}_{BB'}\\ &= \frac{1}{2}\left\{\begin{bmatrix}-[\hat{\boldsymbol{\omega}}\times]-[\delta\boldsymbol{\omega}\times] & \hat{\boldsymbol{\omega}}+\delta\boldsymbol{\omega}\\ -\hat{\boldsymbol{\omega}}^T-\delta\boldsymbol{\omega}^T & 0\end{bmatrix} + \begin{bmatrix}-[\hat{\boldsymbol{\omega}}\times] & -\hat{\boldsymbol{\omega}}\\ \hat{\boldsymbol{\omega}}^T & 0\end{bmatrix}\right\}\mathbf{q}_{BB'}\end{aligned}$$

Therefore,

$$\delta\dot{\mathbf{q}}_{BB'} = \frac{1}{2}\begin{bmatrix}-2[\hat{\boldsymbol{\omega}}\times]-[\delta\boldsymbol{\omega}\times] & \delta\boldsymbol{\omega}\\ -\delta\boldsymbol{\omega}^T & 0\end{bmatrix}\mathbf{q}_{BB'}$$

■

The proof follows the paper by Bani Younes A, Mortari D. *Derivation of All Attitude Error Governing Equations for Attitude Filtering and Control*, Sensors (Basel). 2019;19(21):4682. Published 2019 Oct 28. doi:10.3390/s19214682.